

Exercise 99

If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2}$$

Solution

Use the chain rule to differentiate y .

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(y) \\ &= \frac{d}{dx}[f(u)] \\ &= f'(u) \cdot \frac{d}{dx}(u) \\ &= \frac{df}{du} \frac{du}{dx} \end{aligned}$$

Find the second derivative now.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dx} \left(\frac{df}{du} \frac{du}{dx} \right) \\ &= \left[\frac{d}{dx} \left(\frac{df}{du} \right) \right] \frac{du}{dx} + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \left[\frac{du}{dx} \frac{d}{du} \left(\frac{df}{du} \right) \right] \frac{du}{dx} + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \left[\frac{d}{du} \left(\frac{df}{du} \right) \right] \left(\frac{du}{dx} \right)^2 + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \frac{d^2f}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{df}{du} \frac{d^2u}{dx^2} \\ &= \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2} \end{aligned}$$